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OPEN

DOI 10.1515/tmj-2017-0113

Algebra and local presentability: how algebraic are they? (A survey)

Jiří Adámek[1] and Jiří Rosický[2,*]

[1]Department of Mathematics, Faculty of Electrical Engineering, Czech Technical University in Prague, Czech Republic
[2]Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, 611 37 Brno, Czech Republic
E-mail: j.adamek@tu-bs.de[1], rosicky@math.muni.cz[2]

Dedicated to the eightieth birthday of two mathematicians we admire: Peter Freyd and Bill Lawvere.

Abstract

This is a survey of results concerning the algebraic hulls of two 2-categories: **VAR**, the 2-category of finitary varieties, and **LFP**, the 2-category of locally finitely presentable categories.

2010 Mathematics Subject Classification. **18C10**. 18C35, 08C05
Keywords. variety, algebraic theory, algebraic functor, duality, locally finitely presentable category.

1. Introduction

It is with great pleasure that we recall the years of our close collaboration with Bill Lawvere and present here a survey of the results that we and subsequently other authors obtained, initiated by Bill. He asked: what algebraic properties (a) varieties and (b) locally finitely presentable categories have? Are the corresponding 2-categories algebraic? If not, we ask what their algebraic hull is.

What do we mean by algebraic hulls? Firstly, both varieties and locally finitely presentable categories form a 2-category in a natural sense as we explain in Sections 3 and 4: the category **VAR** has algebraic functors as 1-cells and natural transformations as 2-cells. And the category **LFP** has finitary right adjoints as 1-cells and natural transformations as 2-cells. Next, we follow two passages in analogy to what algebraic hull means in the classical General Algebra.

The first passage is to define equations and their satisfaction by a category. The algebraic (or equational) hull of **VAR** then consists of those categories that satisfy all equations satisfied by every variety. Analogously for **LFP**. We follow this passage in Sections 3 and 4.

The latter passage works with pseudomonads $\mathbb{T}$ on the 2-category **CAT** of categories. The category $\mathbf{CAT}^{\mathbb{T}}$ of algebras is concrete: we have the obvious 2-forgetful functor $U^{\mathbb{T}} : \mathbf{CAT}^{\mathbb{T}} \to \mathbf{CAT}$. The algebraic hull of **VAR** is a concrete 2-embedding $E : \mathbf{VAR} \to \mathbf{CAT}^{\mathbb{T}}$ (meaning that $U^{\mathbb{T}} \cdot E$ is the inclusion 2-functor) universal among all such concrete 2-embeddings. Analogously for **LFP**. This passage is followed in Sections 5-7.

The last Section 8 discusses infinitary varieties and locally λ-presentable categories. And the concept of limits distributing over colimits is recalled in the Appendix.

*The second author is supported by the Grant agency of the Czech republic under the grant P201/12/G028.

Tbilisi Mathematical Journal 10(3) (2017), pp. 279–295.
Tbilisi Centre for Mathematical Sciences.
Received by the editors: 28 September 2017.
Accepted for publication: 02 December 2017.

2. Algebraic duality

The first question to turn to was: what are the canonical morphisms between varieties (as objects)? We were quite surprised not to find an answer in the literature. Varieties can be viewed, in the light of the classical results of Lawvere, see [20], [21], as categories $\mathrm{Mod}\,\mathcal{T}$ of models of algebraic theories $\mathcal{T}$. Here $\mathcal{T}$ is a small category with finite products and $\mathrm{Mod}\,\mathcal{T}$ the category of set-valued functors on $\mathcal{T}$ preserving finite products. (Thus "variety" means a many-sorted, finitary equational class of algebras.) Morphisms of algebraic theories are obvious: the functors between them preserving finite products. They induce functors between varieties preserving

1. limits
2. filtered colimits, and
3. regular epimorphisms.

And vice versa: every functor preserving (1)–(3) is naturally isomorphic to one induced by a morphism of theories. We call these functors *algebraic* (in [3] we used the name algebraically exact). We thus obtain the 2-category

$$\mathbf{VAR}$$

of varieties, algebraic functors and natural transformations. This compares to the 2-category

$$\mathbf{Th}$$

of algebraic theories, their morphisms and natural transformations as follows: we have a 2-functor

$$\mathrm{Mod}\colon \mathbf{Th} \to \mathbf{VAR}^{\mathrm{op}}$$

assigning to every theory $\mathcal{T}$ the variety $\mathrm{Mod}\,\mathcal{T}$ and to every theory morphism $F\colon \mathcal{T} \to \mathcal{T}'$ the algebraic functor $(-)\cdot F\colon\ \mathrm{Mod}\,\mathcal{T}' \to \mathrm{Mod}\,\mathcal{T}$ it induces.

The first attempt at the algebraic duality might be to claim that Mod is an equivalence of categories. But it is not! Consider $\mathbf{Set}$ as a (trivial) variety whose algebraic theory $\mathcal{T}_{\mathbf{Set}}$ is the dual of natural numbers. Every endofunctor of $\mathbf{Set}$ naturally isomorphic to Id is certainly algebraic – and there is a proper class of such pairwise distinct endofunctors. But there is only a countable set of theory endomorphisms of $\mathcal{T}_{\mathbf{Set}}$. Thus, we need to employ the concept of *biequivalence* between 2-categories. Recall that a bifunctor $F\colon \mathcal{K} \to \mathcal{L}$ is a biequivalence provided that

(a) F induces, for every pair K_1, K_2 of objects of $\mathcal{K}$, an equivalence of categories $\mathcal{K}(K_1, K_2) \simeq \mathcal{L}(FK_1, FK_2)$

and

(b) every object of $\mathcal{L}$ is equivalent to FK for some $K \in \mathcal{K}$.

For example, Mod is a biequivalence. This leads us to the following

2.1. Algebraic Duality Theorem (see [3]). The 2-category of varieties is dually biequivalent to the 2-*category of algebraic theories.*

We do not include further details of this duality since the reader can find them in [3] or [9]. See also the paper of Centazzo and Vitale [12] where a unified proof of this and other dualities is presented. Let us mention that the condition (3) above can be substituted by preservation of reflexive coequalizers: These are coequalizers of parallel pairs which are split epis having a joint splitting.

Lemma 2.2. ([3]) Every algebraic functor between varieties preserves reflexive coequalizers.

It is also true that conditions (2) and (3) can be put together by saying that the functor preserves *sifted colimits*. These are colimits of diagrams whose domain $\mathcal{D}$ is a small sifted category, which means that $\mathcal{D}$-colimits in **Set** commute with finite products. Besides filtered colimits, also reflexive coequalizers are sifted. And this is all in the sense that a functor between cocomplete categories preserves sifted colimits iff it preserves filtered colimits and reflexive coequalizers, see [7]. Analogously to the Grothendieck's free completion of a category under filtered colimits, denoted by

$$Ind\,\mathcal{K}$$

see [10] we introduced in [7] the following

2.3. Notation. The free completion of a category $\mathcal{K}$ under sifted colimits is denoted

$$Sind\,\mathcal{K}$$

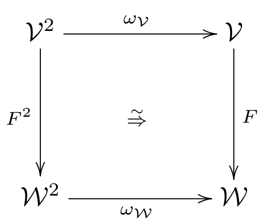
2.4 Proposition. ([7]) For every category $\mathcal{K}$ with finite coproducts $Sind\,\mathcal{K}$ is obtained by completing $Ind\,\mathcal{K}$ freely under reflexive coequalizers.

Corollary 2.5. Algebraic functors between varieties are precisely the functors preserving limits and sifted colimits.

3. How algebraic is algebra?

Finitary algebra, represented by the 2-category **VAR**, is non-fully embedded into the 2-category **CAT** of categories. Is algebra algebraic? That is, is the forgetful 2-functor $U\colon \mathbf{VAR} \hookrightarrow \mathbf{CAT}$ 2-monadic? Or at least pseudomonadic? It is not because U does not preserve U-split coequalizers, see Example 5.1 below. Lawvere suggested to study the "algebraic hull" of the above embedding. This requires defining what an "operation" on **VAR** means and which "equations" these operations satisfy. The algebraic hull then consists of all categories on which those operations are also defined, and satisfy all the equations. The 1-cells of the hull are the functors preserving the given operations, and the 2-cells are the natural transformations.

Before defining a k-ary operation on **VAR**, let us take an example: binary products. Every variety $\mathcal{V}$ has binary products, which defines a functor $\omega_{\mathcal{V}}\colon \mathcal{V}^2 \to \mathcal{V}$ (by choosing a product for every pair). Every algebraic functor $F\colon \mathcal{V} \to \mathcal{W}$ preserves binary products (non-strictly, of course). This means that we obtain a natural isomorphism between $F \cdot \omega^{\mathcal{V}}$ and $\omega^{\mathcal{W}} \cdot F^2$:



Thus, binary products form a binary operation on **VAR** in the following sense:

Definition 3.1. Given a category $\mathcal{K}$, by a *$\mathcal{K}$-ary operation* ω on **VAR** is meant a pseudonatural transformation

$$\omega\colon U^{\mathcal{K}} \Rightarrow U\,.$$

It assigns to every variety $\mathcal{V}$ a functor $\omega_{\mathcal{V}}\colon \mathcal{V}^{\mathcal{K}} \to \mathcal{V}$ and to every algebraic functor $F\colon \mathcal{V} \to \mathcal{W}$ a natural isomorphism

$$\begin{array}{ccc} \mathcal{V}^{\mathcal{K}} & \xrightarrow{\ \omega_{\mathcal{V}}\ } & \mathcal{V} \\ {\scriptstyle F\cdot -}\downarrow & \overset{\omega_F}{\Rightarrow} & \downarrow{\scriptstyle F} \\ \mathcal{W}^{\mathcal{K}} & \xrightarrow[\ \omega_{\mathcal{W}}\ ]{} & \mathcal{W} \end{array}$$

such that $\omega_{G\cdot F} = (G * \omega_F)\cdot(\omega_G * F)$ and $\omega_{Id} = Id$.

Examples are

1. *k-lim*, limits over a small category k,
2. *k-colim*, colimits over a small, filtered category k,
3. *reflexive coequalizers*, an operation of the following arity

$$r \equiv \bullet \ \underset{u_2}{\overset{u_1}{\underset{\xleftarrow{\ d\ }}{\overset{\longrightarrow}{\longrightarrow}}}}\ \bullet \tag{3.1}$$

   with $u_1 \cdot d = \mathrm{id} = u_2 \cdot d$, see Lemma 2.2

and

1. *projections*: for every object i of a category $\mathcal{K}$ define the $\mathcal{K}$-ary operation ω^i by $F \mapsto F(i)$ for all $F\colon \mathcal{K} \to \mathcal{V}$.

Composite Operations. Given categories $\mathcal{K}$ and $\mathscr{L}$, by an *$\mathscr{L}$-tuple of operations* of arity $\mathcal{K}$ is meant a pseudonatural transformation from $U^{\mathcal{K}}$ to $U^{\mathscr{L}}$. This leads to the concept of composing an $\mathscr{L}$-tuple γ of $\mathcal{K}$-ary operations with one $\mathscr{L}$-ary operation ω: the result is one $\mathcal{K}$-ary operation obtained as the following composite

$$U^{\mathcal{K}} \overset{\gamma}{\Rightarrow} U^{\mathscr{L}} \overset{\omega}{\Rightarrow} U$$

of pseudonatural transformations.

We may ask (based on the intuition of the above algebraic duality) whether the above operations are exhaustive: can every operation on **VAR** be composed from the operations of types (1)–(4)? The answer is no, see Example 6.9 below. The reason is that all of the above operations have the following property:

Definition 3.2. An operation ω on **VAR** is *ranked* if its arity is a small category and there exists an infinite regular cardinal λ (the rank of ω) such that every functor $\omega_{\mathcal{V}}$ preserves λ-filtered colimits.

Thus k-lim is ranked: choose λ larger than the number of morphisms of k. And k-colim has rank ω. Since composites of ranked operations are clearly ranked, we can reduce our question about (1)–(4) being exhaustive. Here the answer is affirmative:

Theorem 3.3 ([4, 4.2])**.** All ranked operations on **VAR** are composites of operations of types the (1)–(4) above.

Equations. Let us again start by a trivial example: the binary operations "product" is commutative. This means that for $\delta\colon U^2 \Rightarrow U^2$ swapping the components, we have an equality $\omega \cdot \delta = \omega$ - but, of course, just a nonstrict one. More precisely, we have an invertible modification from $\omega \cdot \delta$ to ω.

Definition 3.4. By an *equation* e over **VAR** between operations ω and ω' of the same arity $\mathcal{K}$ is meant an invertible modification from ω to ω'.

To every variety $\mathcal{V}$ this assigns a natural isomorphism $e_{\mathcal{V}}\colon \omega_{\mathcal{V}} \to \omega'_{\mathcal{V}}$ such that for every algebraic functor $F\colon \mathcal{V} \to \mathcal{W}$ a natural isomorphism

$$\begin{array}{ccc} F\omega_{\mathcal{V}} & \xrightarrow{\;\widehat{\omega}_F\;} & \omega_{\mathcal{W}}F^{\mathcal{K}} \\ {\scriptstyle Fe_{\mathcal{V}}}\downarrow & \overset{\sim}{\Rightarrow} & \downarrow{\scriptstyle e_{\mathcal{W}}F^{\mathcal{K}}} \\ F\omega_{\mathcal{W}} & \xrightarrow[\;\widehat{\omega}'_F\;]{} & \omega'_{\mathcal{W}}F^{\mathcal{K}} \end{array}$$

is given subject to the expected coherence conditions.

Examples of equations are abundant:

(a) Filtered colimits commute over finite limits.

(b) Filtered colimits *distribute* over (infinite) products, see Appendix.

(c) Regular epimorphisms are stable under pullback.

In order to summarize those conditions, we introduced the concept of an algebraically exact category in [4]. If $\mathcal{K}$ has sifted colimits then we have the functor

$$\mathrm{colim} : \mathit{Sind}\,\mathcal{K} \to \mathcal{K}$$

which computes sifted colimits in $\mathcal{K}$, unique up to natural isomorphism.

Definition 3.5. A complete category with sifted colimits is called *algebraically exact* if sifted colimits distribute over limits (see Appendix), i.e., the above functor colim preserves limits.

Every variety is algebraically exact, more generally, every essential localization of a variety is, see [8] 3.3. Among categories with a small regular generator the inverse holds: every algebraically exact category is an essential localization of a variety. In [4] we proved that a category $\mathcal{K}$ with limits und sifted colimits satisfies all the equations between ranked operations holding in **VAR** iff it is algebraically exact. And we asked whether this is equivalent to having the properties (a)-(c) above and being Barr-exact (that is, a regular category with effective equivalence relations). Richard Garner answered this affirmatively by a beautiful application of topos theory:

Theorem 3.6 ([16])**.** A category with limits, filtered colimits, and reflexive coequalizers is algebraically exact iff it is Barr-exact and has properties (a)–(c).

This explains in which sense the properties (a)–(c) summarize *all* equational properties of varieties (w.r.t. ranked operations), since we proved the following result.

Theorem 3.7 ([4], Corollary 4.4)**.** The algebraic hull of **VAR** w.r.t. ranked operations is the 2-category **ALG** of

algebraically exact categories,
functors preserving limits and sifted colimits

and

natural transformations.

We indicate the proof in Section 7.

Open Problem. What is the algebraic hull of **VAR** w.r.t.
(a) all operations of small arities?
(b) all operations?

4. How algebraic is local presentability?

We now take a broader view by considering not only varieties, but all locally finitely presentable (lfp) categories. These are the cocomplete categories $\mathcal{K}$ with a small full subcategory $\mathcal{K}_{\mathrm{fp}}$ which represents all finitely presentable objects and is colimit dense, i.e., every object is a colimit of objects in $\mathcal{K}_{\mathrm{fp}}$. The role that an algebraic theory plays for a variety is played by the dual of $\mathcal{K}_{\mathrm{fp}}$, $\mathcal{T} = (\mathcal{K}_{\mathrm{fp}})^{\mathrm{op}}$. And where $\mathrm{Mod}\,\mathcal{T}$ was the category of set-valued functors preserving finite products, we now take the following category

$$\mathrm{Lex}\,\mathcal{T} = \text{ functors in } \mathbf{Set}^{\mathcal{T}} \text{preserving finite limits.}$$

Indeed, every lfp category $\mathcal{K}$ is equivalent to $\mathrm{Lex}\,\mathcal{K}_{\mathrm{fp}}^{\mathrm{op}}$, see [17].

What are the canonical morphisms between lfp categories as objects? The answer is given by the well-known duality due to Gabriel and Ulmer. The above assignment $\mathcal{T} \mapsto \mathrm{Lex}\,\mathcal{T}$ defines a 2-functor

$$\mathrm{Lex}\colon\ \mathbf{Cat}_{lex} \to \mathbf{CAT}^{\mathrm{op}}$$

from the 2-category $\mathbf{Cat}_{\mathrm{lex}}$ of small, finitely complete categories, lex functors (i.e., preserving finite limits), and natural transformations. To every lex functor $F\colon k \to k'$ it assigns the functor $(-)\cdot F\colon\ \mathrm{Lex}\,k' \to \mathrm{Lex}\,k$.

Gabriel-Ulmer Duality ([17])**.** Lex is a dual biequivalence between $\mathbf{Cat}_{\mathrm{lex}}$ and the 2-category **LFP** of

locally finitely presentable categories,
functors preserving limits and filtered colimits

and

natural transformations.

Is **LFP** algebraic? That is, is the forgetful functor

$$V\colon \mathbf{LFP} \to \mathbf{CAT}$$

monadic in an appropriate sense? The answer is negative (for the same reason why it is negative for **VAR**), see Section 5. Thus we ask again: what is the algebraic hull of **LFP**?

Operations of arity $\mathcal{K}$ are now pseudonatural transformations from $V^{\mathcal{K}}$ to V. Composite operations and equations are introduced analogously to **VAR**.

Here we are more lucky. Under a certain set-theoretical axiom (R), see 4.2 below, we have a simple description of all operations on **LFP** of small arities: they are precisely the composites of the following types

(1) k-lim for small categories k,

(2) k-colim for small filtered categories k, and

(3) projections.

And all equational properties of these operations can be derived from the following ones:

(a) finite limits commute with filtered colimits, and

(b) products distribute over filtered colimits, see Appendix.

This leads to a concrete description of the algebraic hull of **LFP** w.r.t. operations of small arities: it is formed by all precontinuous categories. They are defined below where, analogously to $Sind$, for every category $\mathcal{K}$ with filtered colimits we denote by

$$\text{colim} : Ind\mathcal{K} \to \mathcal{K}$$

the functor computing filtered colimits in $\mathcal{K}$.

Definition 4.1. A complete category $\mathcal{K}$ with filtered colimits is called

(a) *continuous* if colim : $Ind\mathcal{K} \to \mathcal{K}$ has a left adjoint, and

(b) *precontinuous* if colim preserves limits.

The first concept was introduced by Johnstone and Joyal [18] where they prove in 2.4 that every lfp category is continuous. The latter concept is from [5]. Every essential localization of an lfp category is precontinuous, and among categories with a regular generator, these are all the precontinuous categories, see [8], 2.7.

Remark 4.2. The following axiom was proved by Donder [14] to be consistent with set theory:

(R) Every uniform ultrafilter is regular.

That is, given an ultrafilter $\mathcal{U}$ all members of which have the same cardinality k, there exists a subset $\mathcal{V} \subseteq \mathcal{U}$ of cardinality k such that every member of $\mathcal{U}$ meets only finitely many members of $\mathcal{V}$.

Theorem 4.3 ([5, 6.6])**.** Assuming (R) the algebraic hull of **LFP** w.r.t. operations of small arities is the 2-category of precontinuous categories, functors preserving limits and filtered colimits, and natural transformations.

5. Algebraic and continuous lattices

In this short section we explain why neither **VAR** nor **LFP** is pseudomonadic, and we present our proof, from [5], that the algebraic hull of algebraic lattices is formed by the continuous ones. This was originally proved by Day [13] and Wyler [25] and the reason for our alternative proof below is that the procedure is analogous to that we use in Section 6 for **LFP** and in Section 7 for **VAR**.

An *algebraic lattice* is a poset that is lfp (or a variety) as a category: a complete lattice in which every element is a filtered joins of compact, i.e. finitely presentable, elements. We consider algebraic lattices as infinitary algebras whose k-ary operations are k-meets (for all discretely ordered sets k) and k-joins (for all directed posets k). This class of algebras is not equationally presentable because it is not closed under quotients. That is, if

$$\mathbf{Alg}$$

denotes the category of algebraic lattices and functions preserving meets and directed joins, the forgetful functor

$$W\colon\ \mathbf{Alg} \to \mathbf{Pos}$$

is not monadic: it does not preserve W-split coequalizers. We illustrate this in the next example, denoting by

$$\mathit{Ind}\, L$$

the free completion of a poset L under filtered joins. Let us recall that if L has finite meets, then $\mathit{Ind}\, L$ is an algebraic lattice and has the following concrete description:

$$\mathit{Ind}\, L = \ \text{all directed } \downarrow\text{-sets of } \ L \quad (\text{ordered by } \subseteq)$$

Example 5.1. Let L be an algebraic lattice with a non-algebraic quotient

$$e\colon L \to L/\mathrm{E}\ .$$

Here $\mathrm{E} \subseteq L \times L$ is a congruence for which the quotient map e preserves meets and directed joins. The projections u_1, $u_2\colon\ \mathrm{E} \to L$ have unique extensions

$$\mathit{Ind}\, L \overset{\overline{u}_1}{\underset{\overline{u}_2}{\rightrightarrows}} L$$

to morphisms of **Alg** which clearly form a W-split pair. That is, the coequalizer of $W\overline{u}_1$ and $W\overline{u}_2$ splits in **Pos**. Since e is the coequalizer of $\overline{u}_1$, $\overline{u}_2$ in **Pos**, we conclude that W does not preserve W-split coequalizers.

Remark 5.2. What does the above example say us about the 2-functor $U\colon$ **VAR** $\to$ **CAT**? That U also fails to preserve U-split coequalizers. Now this means, by Beck Theorem, that U is not monadic. However, we need more: the appropriate concept we need (for dealing with constructions such as free completion under filtered colimits) is *pseudomonad*, a non-strict variant of a 2-monad. Pseudomonads were introduced by Day and Street [15], and the corresponding Beck Theorem for them was proved by Le Creurer, Marmolejo and Vitale [22].

Corollary 5.3. Neither $U\colon$ **VAR** $\to$ **CAT** nor $V\colon$ **LFP** $\to$ **CAT** is pseudomonadic.

But let us return to the "posetal shadow", the category of algebraic lattices. The above completion under filtered colimits defines a monad *Ind* on **Pos**. Furthermore denote by *Meet* L the free completion under meets. This defines a monad *Meet* on **Pos** (assigning to L the poset of all $\uparrow$-sets ordered by the opposite of $\subseteq$).

5.4 Proposition ([5, 2.5])**.** The monad *Meet* distributes over *Ind* since
(a) if L is a complete meet-semilattice, so is *Ind* L, and
(b) if $f\colon L \to L'$ is a meet-semilattice homomorphism,
then so is *Ind* f.
Consequently, there is a monad structure on the composite

$$\mathit{Ind} \circ \mathit{Meet}\,.$$

Corollary 5.5. Since the free algebras of the above monad,

$$\mathit{Ind}\,(\mathit{Meet}\,L)$$

are algebraic lattices, due to *Meet* L having (finite) joins, it follows that the category

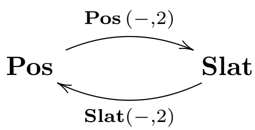
$$\mathbf{Pos}^{\,(\mathit{Ind}\circ\mathit{Meet})}$$

of algebras for the monad *Ind* $\circ$ *Meet* is the algebraic hull of **Alg**. This is precisely the category of continuous lattices.

Example 5.6. Another approach to the above corollary is to consider the two-element chain 2 as a dualizing object for the categories **Pos** and

$$\mathbf{Slat} = \text{meet semilattices and their homomorphisms.}$$

We obtain an adjoint situation (enriched over **Pos**) as follows



yielding a monad on **Pos** denoted by

$$\mathbb{D}\colon L \mapsto \mathbf{Slat}(2^L, 2).$$

It assigns to every poset the set of all filters on it. It was already observed by Day [13] that continuous lattices form the category of Eilenberg-Moore algebras for the filter monad.

5.7. Fact (see [5]). The monad $\mathbb{D}$ is isomorphic to the monad *Ind* $\circ$ *Meet*.

6. The algebraic hull of LFP

We first describe the algebraic hull of **LFP** w.r.t. all operations (of possibly large arities) by introducing a pseudomonad $\mathbb{D}^*$ analogous to $\mathbb{D}$ in Example 5.6. (Well, "describe" is a bit optimistic: we will show that the Eilenberg-Moore category of $\mathbb{D}^*$ is the algebraic hull of **LFP**, but we have no concrete characterization of the algebras for $\mathbb{D}^*$.) Then we turn to operations of small arities on **LFP** and prove that

(a) they are all generated by small limits and small filtered colimits,

(b) the 2-category of precontinuous categories (see 4.1) is the algebraic hull of **LFP** w.r.t. operations of small arities -assuming the axiom (R),

and

(c) without (R), precontinuous categories form the algebraic hull of **LFP** w.r.t. ranked operations (see 3.2).

6.1. Notation. For the 2-categories **CAT** (of all categories) and $\mathbf{CAT}_{\text{lex}}$ (of all lex categories) we use their joint object **Set** as a dualizer and consider the enriched functor

$$\mathbf{Cat}(-,\mathbf{Set}):\ \mathbf{CAT} \to \mathbf{CAT}_{\text{lex}}^{\text{op}}$$

as a left adjoint to

$$\mathbf{Cat}_{\text{lex}}(-,\mathbf{Set}):\ \mathbf{CAT}_{\text{lex}}^{\text{op}} \to \mathbf{CAT}\,.$$

The corresponding pseudomonad

$$\mathbb{D}^*$$

on **CAT** assigns to every category $\mathcal{K}$ the category

$$\mathbb{D}^*\mathcal{K} = \text{lex functors } \mathbf{Set}^{\mathcal{K}} \to \mathcal{K}$$

to every functor $F\colon \mathcal{K} \to \mathscr{L}$ the functor

$$\mathbb{D}^*F = (\ ) \cdot F\colon \mathbf{Set}^{\mathscr{L}} \to \mathbf{Set}^{\mathcal{K}}$$

and to every natural transformations f the natural transformation $\left(\mathbb{D}^*f\right)_H = H * \mathbf{Set}^{f}$.

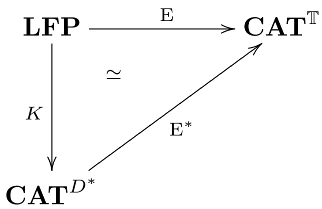
Recall from Introduction that he algebraic hull of **LFP** is given by a concrete 2-embedding of **LFP** into an Eilenberg-Moore category over **CAT** with the expected universal property which we make explicit below.

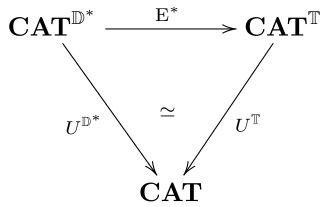
Theorem 6.2 ([5, 4.6])**.** The Eilenberg-Moore 2-category of $\mathbb{D}^*$,

$$\mathbf{CAT}^{\mathbb{D}^*}$$

is the algebraic hull of **LFP** (w.r.t. the comparison 2-functor $K\colon \mathbf{LFP} \to \mathbf{CAT}^{\mathbb{D}^*}$).

That is, for every pseudomonad $\mathbb{T}$ on **CAT** and every concrete 2-embedding $E\colon \mathbf{LFP} \to \mathbf{CAT}^{\mathbb{T}}$ there is an extension of E to a concrete pseudofunctor $\mathrm{E}^*\colon \mathbf{CAT}^{\mathbb{D}^*} \to \mathbf{CAT}^{\mathbb{T}}$. We thus have the following natural isomorphisms:





6.3. Notation. The "small core" of the pseudomonad $\mathbb{D}^*$ is given by forming, for every category $\mathcal{K}$, the colimit

$$\mathbb{D}^*_{\mathrm{small}}\mathcal{K} = \operatorname{colim} \mathbb{D}^* k$$

of the diagram of all free $\mathbb{D}^*$-algebras on k, where k ranges over all small subcategories of $\mathcal{K}$.

It is easy to see that this yields a sub-pseudomonad $\mathbb{D}^*_{\mathrm{small}}$ of $\mathbb{D}^*$. And the algebraic hull of **LFP** w.r.t. small-arity operations is the Eilenberg-Moore 2-category for $\mathbb{D}^*_{\mathrm{small}}$. Fortunately, we have a nice description of $\mathbb{D}^*_{\mathrm{small}}$. This is based on the following analogy of Proposition 5.4:

Definition 6.4. (a) We denote by

$$\mathit{Lim}$$

the pseudomonad on **CAT** of free completion under small limits. Thus for k small we have

$$\mathit{Lim}\ k = \left(\mathbf{Set}^{\,k}\right)^{\mathrm{op}}$$

w.r.t. the Yoneda embedding of k.

(b) We denote by

$$\mathit{Ind}$$

the pseudomonad on **CAT** of free completion under filtered colimits. For every finitely cocomplete category $\mathscr{L}$ we have

$$\mathit{Ind}\ \mathscr{L} = \mathbf{Cat}_{\mathrm{lex}}\left(\mathscr{L}^{\mathrm{op}}, \mathbf{Set}\right).$$

6.5 Proposition ([5, 5.5])**.** The pseudomonad *Lim* distributes over *Ind* because

(a) if $\mathscr{L}$ is a complete category, so is $\mathit{Ind}\,\mathscr{L}$,

and

(b) if $F\colon \mathscr{L} \to \mathscr{L}'$ preserves limits ($\mathscr{L}$ and $\mathscr{L}'$ complete), so does $\mathit{Ind}\,F$.

Corollary 6.6. $\mathit{Ind} \circ \mathit{Lim}$ carries the structure of a pseudomonad on **CAT**.

Remark 6.7. In order to obtain a concrete characterization of the algebraic hull of **LFP** w.r.t. operations of small arities, we would like to prove that the pseudomonads $\mathbb{D}^*_{\mathrm{small}}$ and $\mathit{Ind} \circ \mathit{Lim}$ are biequivalent. This implies that every operation of small arity is composable from small limits, small colimits, and projections - thus it has a rank. This is not true absolutely, as the next example demonstrates. But it is true assuming (R), see Remark 4.2, as we explain after that example. Recall that a functor is called *accessible* if it preserves λ-filtered colimits for some regular cardinal λ.

Example 6.8 (Reiterman [23])**.** Assume the existence of arbitrarily large measurable cardinals. That is, for every cardinal k there exists a non-trivial ultrafilter $\mathcal{F}_k$ (on a set, say, M_k) closed under meets of k members. Then a lex endofunctor H of **Set** which is not accessible can be constructed as follows. For every cardinal k let Q_k be the quotient of the hom-functor $\mathbf{Set}\,(M_k,-)$ merging two maps from M_k iff they agree on some set $Y \in \mathcal{F}_k$. Define a transfinite chain H_k $(k \in \mathrm{Ord}\,)$ of set functors by $H_0 = Id$, $H_{k+1} = Q_k \circ H_k$ and $H_k = \operatorname*{colim}_{l<k} H_l$ for limit ordinals k. Then the transfinite colimit $H = \operatorname*{colim}_{k \in \mathrm{Ord}} H_k$ exists in $[\mathbf{Set}\,,\mathbf{Set}\,]$, is lex but not accessible.

Example 6.9. A unary operation without rank on **LFP** and on **VAR**.

Define a unary operation ω on **VAR** by using the functor H of the above Example as follows: for every variety $\mathcal{V} = \mathrm{Mod}\,\mathcal{T}$ define the functor

$$\omega_{\mathcal{V}} : \mathcal{V} \to \mathcal{V}$$

by post-composition with H, that is, $\omega_{\mathcal{V}}(F) = H \cdot F$ for all $F\colon \mathcal{T} \to \mathbf{Set}$ in $\mathrm{Mod}\,\mathcal{T}$. Since H does not preserve λ-filtered colimits for any λ, we see that neither $\omega_{\mathbf{Set}}$ preserves them. Thus ω is not ranked.

Analogously for **LFP**: here we define $\omega_{\mathcal{V}}$, for $\mathcal{V} = \mathrm{Lex}\,\mathcal{T}$, by post-composition with H.

Theorem 6.10 ([1])**.** Assuming (R), every lex functor from $\mathbf{Set}^{\,k}$ to **Set** , with k small, is accessible.

Corollary 6.11 ([5, 5.8])**.** Assuming (R), the pseudomonads $\mathbb{D}^*_{\mathrm{small}}$ and $Ind \circ Lim$ are biequivalent.

Now the Eilenberg-Moore category of $Ind \circ Lim$ is easy to describe: It is the 2-category of precontinuous categories. That is, the above corollary states precisely what Theorem 4.3 does. Without (R), the weakening of Theorem 4.3, where only ranked operations are used, holds, see Theorem 7.5 below.

7. The algebraic hull of VAR

The situation with **VAR** is analogous to (but a bit less clear than) **LFP**: here also two pseudomonads on **CAT** are composed and the algebras of the resulting pseudomonads form the algebraic hull w.r.t. ranked operations on **LFP**. The one missing step is the result involving (R): we do not know whether (R) implies that all operations of small arities are ranked.

The role that filtered colimits and the pseudomonad *Ind* play for **LFP** is played by sifted colimits here (see 2.3).

7.1. Notation. Recall that $Sind\,\mathcal{K}$ denotes the free completions of a category $\mathcal{K}$ under sifted colimits. For small categories with finite coproducts k we have, as proved in [7]

$$Sind\,k = \mathrm{Mod}(k^{\mathrm{op}})\,. \tag{7.1}$$

We denote by

$$Sind$$

the pseudomonad of free completion under sifted colimits on **CAT**.

Theorem 7.2 ([5, 3.11])**.** The pseudomonad *Lim* distributes over *Sind* because
(a) if $\mathscr{L}$ is a complete category, so is $\mathit{Sind}\,\mathscr{L}$,
and
(b) if $F\colon \mathscr{L} \to \mathscr{L}'$ preserves limits ($\mathscr{L}$ and $\mathscr{L}'$ complete), so does $\mathit{Sind}\,F$.

Corollary 7.3. $\mathit{Sind} \circ \mathit{Lim}$ carries the structure of a pseudomonad on **CAT**.

The Eilenberg-Moore category of the pseudomonad $\mathit{Sind} \circ \mathit{Lim}$ is easily seen to be the 2-category **ALG** of algebraicaly exact categories, see 3.7. We now want to explain in which sense the Eilenberg-Moore 2-category is the algebraic hull of **VAR**. Its free algebra on a small category k is, since $\mathit{Lim}\, k = (\mathbf{Set}^{\,k})^{\mathrm{op}}$, the category

$$\mathit{Sind}\left(\mathbf{Set}^{\,k}\right)$$

which, unfortunately, does not have the form $\mathrm{Mod}\,\mathcal{T}$, see (7.1), although $\mathbf{Set}^{\,k}$ has finite coproducts, because $\mathbf{Set}^{\,k}$ is not small. We thus take the essentially small full subcategory of **Set**

$$\mathbf{Set}_{\,\lambda} = \text{all sets of cardinality } \leq \lambda.$$

Then **Set** is a (large) colimit of $\mathbf{Set}_{\,\lambda}$ for $\lambda \in \mathrm{Card}$, and $\mathit{Sind}(\mathbf{Set}^{\,k})$ is a colimit of the following varieties

$$\mathit{Sind}\left(\mathbf{Set}^{\,k}_{\,\lambda}\right) = \mathrm{Mod}\left(\mathbf{Set}^{\,k}_{\,\lambda}\right)^{\mathrm{op}}.$$

Now the fact that we only consider small categories k means that only operations of small arities are taken. And for every λ the operations encoded by $\mathit{Sind}\left(\mathbf{Set}^{\,k}_{\,\lambda}\right)$ are those of rank λ. We thus obtain a direct proof of Theorem 3.7.

Precisely the same procedure can be applied to **LFP**: instead of *Sind* we use *Ind* and consider the free algebras of $\mathit{Ind} \circ \mathit{Lim}$ on small categories k:

$$\mathit{Ind}\,(\mathbf{Set}^{\,k})$$

as colimits of the transfinite chains $\mathit{Ind}\,(\mathbf{Set}^{\,k}_{\,\lambda})$. We obtain the following result (that, by omission, was not formulated in [5]). It does not use any set-theoretic axiom:

Theorem 7.4. The algebraic hull of **LFP** w.r.t. operations with rank is the 2-category of precontinuous categories.

8. Infinitary algebra and local λ-presentability

All results of Section 6 generalize, for infinite regular cardinals λ, without problems from **LFP** to the 2-category

$$\lambda\text{-}\mathbf{LP}$$

of all locally λ-presentable categories, λ-accessible functors preserving limits, and natural transformations. Recall that given a regular cardinal λ, a category is locally λ-presentable if it is complete and has an essentially small full subcategory $\mathcal{K}_\lambda$ on all λ-presentable objects, which is colimit dense. The forgetful functor $V_\lambda\colon \lambda\text{-}\mathbf{LP} \to \mathbf{CAT}$ is not pseudomonadic (for the same reason that V is not). We want to describe the algebraic hull of λ-**LP**. Here operations are pseudonatural transformations from $V_\lambda^{\mathcal{K}}$ to V_λ, and equations are defined completely analogously to the case of **LFP**. If

$$\mathit{Ind}_{\,\lambda}$$

denotes the pseudomonad on **CAT** of free completion under λ-filtered colimits, then the proof that *Lim* distributes over it is completely analogous to the case $\lambda = \omega$ in [5]. This yields a pseudomonad $Lim \circ Ind_\lambda$. Its algebras are the *λ-precontinuous categories* $\mathcal{K}$, i.e. those for which $\operatorname{colim}\colon Ind_\lambda\mathcal{K} \to \mathcal{K}$ preserves limits.

Theorem 8.1. Assuming (R), the algebraic hull of λ-**LP** w.r.t. operations of small arities is the 2-category of λ-precontinuous categories, functors preserving limits and λ-filtered colimits, and natural transformations.

Indeed, the role (R) plays is to make sure that given a small category k every functor $F\colon \mathbf{Set}^k \to \mathbf{Set}$ preserving λ-small limits is accessible – and this follows from Theorem 4.3.

Without (R) the theorem holds for the algebraic hull w.r.t. all ranked operations, analogously to Theorem 7.4.

Also the description of the full algebraic hull generalizes smoothly from **LFP**. Let $\mathbf{CAT}_\lambda$ be the 2-category of all categories having λ-small limits, functors preserving λ-small limits, and natural transformations. The adjoint situation $\mathbf{CAT}(-,\mathbf{Set}) \dashv \mathbf{CAT}_\lambda(-,\mathbf{Set})$ yields a pseudomonad $\mathbb{D}^*_\lambda$ on **CAT** whose Eilenberg-Moore category is the algebraic hull of λ-**LP**.

The situation with λ-ary varieties is more colorful. Denote by

$$\lambda\text{-}\mathbf{VAR}$$

the 2-category of

λ-ary varieties, i.e., equational classes of λ-ary (possibly many-sorted) algebras,

λ-algebraic functors, i.e., functor preserving limits, λ-filtered colimits and regular epimorphisms,

and

natural transformations.

Analogously, we form the 2-category λ-**Th** of λ-ary theories (i.e. small categories with λ-small products), functors preserving λ-small products and natural transformations.

Every variety is equivalent to one of the form $\mathrm{Mod}_\lambda\,\mathcal{T}$ where $\mathcal{T}$ is a λ-ary algebraic theory and $\mathrm{Mod}_\lambda\,\mathcal{T}$ is the category of all functors in $\mathbf{Set}^{\mathcal{T}}$ preserving λ-small products. Thus Mod_λ gives a 2-functor from λ-**Th** to $(\lambda\text{-}\mathbf{VAR})^{\mathrm{op}}$ analogous to Mod above.

λ-Algebraic Duality Theorem ([2]). The 2-functor Mod_λ is a dual biequivalence of λ-**Th** and λ-**VAR**.

However, there is a fundamental catch when trying to generalize algebraic exactness to λ-ary varieties: the expected generalization of sifted categories does not work.

Theorem ([2])**.** Let λ be an uncountable regular cardinal. Then every small category $\mathcal{D}$ such that $\mathcal{D}$-colimits in **Set** commute with λ-small products is λ-filtered.

Thus a direct generalization of results of Section 3 does not work.

Open Problem. Describe the algebraic hull of λ-**VAR** w.r.t. operations with rank.

Appendix

A Limits distributing over colimits

Whereas the concept of a class $\mathbb{L}$ of limits *commuting* over a class $\mathbb{C}$ of colimits is well known (e.g. finite limits commute with filtered colimits in varieties), the distribution of limits over colimits is less often encountered.

Suppose a class $\mathbb{C}$ of small categories is given. Every category $\mathcal{K}$ has a *free completion under* $\mathbb{C}$*-colimits* (i.e., colimits of diagrams $D\colon \mathcal{C} \to \mathcal{K}$ where $\mathcal{C} \in \mathbb{C}$). This is a full embedding

$$\mathrm{E}\colon \mathcal{K} \hookrightarrow \mathit{Colim}_{\mathbb{C}}\,\mathcal{K}$$

into a category with $\mathbb{C}$-colimits with the expected universal property:

> for every category $\mathscr{L}$ with $\mathbb{C}$-colimits we have an equivalence between $[\mathcal{K}, \mathcal{L}]$ and the category of $\mathbb{C}$-colimits preserving functors from $\mathit{Colim}_{\mathbb{C}}\,\mathcal{K}$ to $\mathscr{L}$, given by precomposition with E.

In particular, whenever $\mathcal{K}$ has $\mathbb{C}$-colimits we get a functor

$$\mathrm{colim} : \mathit{Colim}_{\mathbb{C}}\,\mathcal{K} \to \mathcal{K}$$

computing $\mathbb{C}$-colimits in $\mathcal{K}$. It corresponds to $\mathit{Id} \in [\mathcal{K}, \mathcal{K}]$.

Remark A.1. Given another class $\mathbb{L}$ of small categories, to say that $\mathbb{C}$-colimits commute in $\mathcal{K}$ with $\mathbb{L}$-limits means that for every diagram

$$D : \mathcal{C} \times \mathcal{L} \to \mathcal{K}$$

where $\mathcal{C} \in \mathbb{C}$ and $\mathcal{L} \in \mathbb{L}$ the canonical morphism

$$\operatorname*{colim}_{c\in\mathcal{C}} \lim_{l\in\mathcal{L}} \mathcal{D}(c,l) \to \lim_{l\in\mathcal{L}} \operatorname*{colim}_{c\in\mathcal{C}} D(c,l)$$

is invertible.

Definition A.2. ([4]) Let $\mathbb{C}$ and $\mathbb{L}$ be classes of small categories. Given a category $\mathcal{K}$ with $\mathbb{C}$-colimits and $\mathbb{L}$-limits, we say that $\mathbb{C}$-colimits *distribute* over $\mathbb{L}$-limits in $\mathcal{K}$ if the functor

$$\mathrm{colim}\colon\ \mathit{Colim}_{\mathbb{C}}\,\mathcal{K} \to \mathcal{K}$$

preserves $\mathbb{L}$-limits.

Example A.3. Filtered colimits distribute over finite products iff they commute with them. For infinite products distributivity is strictly weaker. For example, in **Set** filtered colimits do *not* commute with countable powers: consider $\mathbb{N}$ as a colimit of $0 \to 1 \to 2 \to \dots$ where $n = \{0,\dots,n{-}1\}$. Then $\mathbb{N}^{\mathbb{N}}$ is strictly larger than $\mathrm{colim}\, n^{\mathbb{N}}$. In contrast:

Example A.4. In every lfp category filtered colimits distribute over products. This was already proved by Artin, Grothendieck and Verdier in [10]. To verify this, recall the following description of $\mathit{Ind}\,\mathcal{K}$ from [18]: objects are all small filtered diagrams $D : \mathcal{D} \to \mathcal{K}$. Morphisms into another filtered diagram $\bar{D} : \bar{\mathcal{D}} \to \mathcal{K}$ are compatible families of equivalence classes $[f_d]_{d\in\mathrm{obj}\,\mathcal{D}}$ of morphisms $f_d : Dd \to \bar{D}d'$, $d' \in \mathrm{obj}\,\bar{\mathcal{D}}$, under the smallest equivalence $\sim$ with $f_d \sim D\bar{u} \cdot f_d$ for every morphism

$\bar{u} : d' \to d''$ in $\bar{\mathcal{D}}$. Compatibility means that for every morphism $v : d_1 \to d_2$ in $\mathcal{D}$ we have $f_{d_1} \sim f_{d_2} \cdot Dv$.

Now a product in $Ind\mathcal{K}$ is easy to describe: given objects $D_i : \mathcal{D}_i \to \mathcal{K}$ for $i \in I$, their product is the filtered diagram

$$D : \prod_{i \in I} \mathcal{D}_i \to \mathcal{K}, \qquad D(d_i)_{i \in I} = \prod_{i \in I} D_i d_i$$

And in every lfp category the colimit of D is canonically given by the product of colimits of D_i, $i \in I_j$, see e.g. [8], proof of 2.1.

Now, distributivity of filtered colimits in $\mathcal{K}$ over products means that $\mathcal{K}$ has both, and that given filtered diagrams D_i $(i \in i)$ we have

$$\operatorname{colim} D = \prod_{i \in I} \operatorname{colim} D_i$$

Example A.5. Filtered colimits in lfp categories actually distribute over *all* limits. This follows from the fact that $\mathcal{K}$ is continuous (see Definition 4.1). However, distributivity of filtered colimits over equalizers is a little bit less intuitive, see the proof of 2.1 in [8].

Example A.6. (a) Distributivity of sifted colimits over products is given by a completely analogous formula to that for filtered colimits, except that here the diagrams are sifted.

(b) In every finitary variety of algebras, sifted colimits distribute over products. Indeed, this follows from algebraic exactness (see 3.7) since this states that in varieties sifted colimits distribute over *all* limits.

Observe that (a) does not hold in lfp categories in general.

Acknowledgement. We are grateful to the referee for suggestions which improved our presentation.

References

[1] J. Adámek, V. Koubek and V. Trnková, *How large are left exact functors?* Theory Appl. Categ. 13 (2001), 377–390.

[2] J. Adámek, V. Koubek and J. Velebil, *A duality between infinitary varieties and algebraic theories*, Comment. Math. Univ. Carolin. 41 (2000), 529–541.

[3] J. Adámek, F. W. Lawvere and J. Rosický, *On the duality between varieties and algebraic theories*, Algebra Universalis 49 (2003), 35–49.

[4] J. Adámek, F. W. Lawvere and J. Rosický, *How algebraic is algebra?*, Theory Appl. Categ. 8 (2001), 253–283.

[5] J. Adámek, F. W. Lawvere and J. Rosický, *Continuous categories revisited*, Theory Appl. Categ. 11 (2003), 252–282.

[6] J. Adámek and J. Rosický, *Locally presentable and accessible categories*, Cambridge Univ. Press 1994.

[7] J. Adámek and J. Rosický, *On sifted colimits and generalized varieties*, Theory Appl. Categ. 8 (2001), 33–53

[8] J. Adámek, J. Rosický and E. M. Vitale, *On algebraically exact categories and essential localizations of varieties*, J. Algebra 244 (2001), 450–477.

[9] J. Adámek, J. Rosický and E. M. Vitale, *Algebraic theories*, Cambridge Univ. Press 2011.

[10] M. Artin, A. Grothendieck and J. L. Verdier, *Théorie des topos et cohomologie étale des schémas*, Lecture Notes in Math. 269, Springer-Verlag, Berlin 1972.

[11] J. Beck, *Distributive laws*, Lecture Notes in Math. 80 (1969), 119–140.

[12] C. Centazzo and E. M. Vitale, *A duality relative to a limit doctrine*, Theory Appl. Categ. 10 (2002), 486–497.

[13] A. Day, *Filter monads, continuous lattices and closure systems*, Canad. J. Math. 27 (1975), 50–59.

[14] H.-D. Donder, *Regularity of ultrafilters and the core model*, Israel J. Math. 63 (1988), 289–322.

[15] B. Day and R. Street, *Monoidal bicategories and Hopf algebroids*, Adv. Math. 129 (1997), 99–157.

[16] R. Garner, *A characterization of algebraic exactness*, J. Pure Appl. Algebra 8 (2017), 1421–1426.

[17] P. Gabriel and F. Ulmer, *Lokal Präsentierbare Kategorien*, Lecture Notes in Math. 221, Springer-Verlag 1971.

[18] P. Johnstone and A. Joyal, *Continuous categories and exponentiable toposes*, J. Pure Appl. Algebra 25 (1982), 255–296.

[19] A. Kock, *Monads for which structures are adjoint to units*, J. Pure Appl. Algebra 104 (1995), 41–59.

[20] F. W. Lawvere, *Functorial semantics of algebraic theories*, Proc. Nat. Acad. Sci. 50 (1963), 865–872.

[21] F. W. Lawvere, Ordinal sums and equational doctrines, Lecture Notes in Math. 80 (1969), 141–155.

[22] I. J. Le Creurer, F. Marmolejo and E. M. Vitale, *Beck's theorem for pseudo-monads*, J. Pure Appl. Algebra 173 (2002), 293–313.

[23] J. Reiterman, *An example concerning set functors*, Comment. Math. Univ. Carolin. 12 (1971), 227–233.

[24] V. Trnková, *On descriptive classification of set functors I*, Comment. Math. Univ. Carolin. 12 (1971), 143–173.

[25] O. Wyler, *Algebraic theories of continuous lattices* In: Continuous lattices (ed. B. Banaschewski and R.-E. Hoffmann), Lecture Notes in Math. 871, 390–413, Springer Verlag, Berlin 1981.